Linear Algebra II

17/03/2016, Thursday, 14:00-16:00

1 (10 + 10 = 20 pts) **Inner product spaces**

Consider the vector space $\mathbb{R}^{2\times 2}$ with the inner product

$$\langle A, B\rangle = \text{trace}(A^T B).$$

Determine the orthogonal complement of

(a) the subspace of all 2×2 diagonal matrices.

(b) the subspace of all 2×2 symmetric matrices.

REQUIRED KNOWLEDGE: **Orthogonality, orthogonal complement.**

SOLUTION:

1a: Let $\mathcal{D}$ denote the subspace of all 2×2 diagonal matrices. Note that

$$A \in \mathcal{D}^\perp \iff \langle D, A\rangle = 0 \text{ for all } D \in \mathcal{D}$$

by the definition of the orthogonal complement. Then, we have

$$\begin{aligned} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{D}^\perp \iff \langle \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle &= \text{trace}(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\ &= \text{trace}(\begin{bmatrix} xa & xb \\ yc & yd \end{bmatrix}) \\ &= xa + yd = 0 \text{ for all } x \text{ and } y. \end{aligned}$$

This means that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{D}^\perp \iff a = 0 \text{ and } d = 0$$

and hence

$$\mathcal{D}^\perp = \{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \mid b, c \in \mathbb{R}\}.$$

1b: Let $\mathcal{S}$ denote the subspace of all 2×2 symmetric matrices. Note that

$$A \in \mathcal{S}^\perp \iff \langle S, A\rangle = 0 \text{ for all } S \in \mathcal{S}$$

by the definition of the orthogonal complement. Then, we have

$$\begin{aligned} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{S}^\perp \iff \langle \begin{bmatrix} x & y \\ y & z \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle &= \text{trace}(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\ &= \text{trace}(\begin{bmatrix} xa + yc & xb + yd \\ ya + zc & yb + zd \end{bmatrix}) \\ &= xa + yc + yb + zd \\ &= xa + y(c + b) + zd = 0 \text{ for all } x,\ y, \text{ and } z. \end{aligned}$$

This means that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{S}^{\perp} \iff a = 0,\, d = 0 \text{ and } b + c = 0$$

and hence

$$\mathcal{S}^{\perp} = \{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \mid b \in \mathbb{R} \}.$$

Consider the matrix

$$M = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}$$

where a, b are real numbers and $b \neq 0$.

(a) Determine if M is diagonalizable without finding its eigenvalues and eigenvectors. Justify your answer.

(b) Find the eigenvalues of M.

(c) Find an orthogonal matrix that diagonalizes M.

REQUIRED KNOWLEDGE: **Eigenvalues, eigenvectors, diagonalization.**

SOLUTION:

2a: The matrix M is symmetric and hence it is diagonalizable.

2b: Note that

$$\det(M - \lambda I) = \det(\begin{bmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{bmatrix}) = (a-\lambda)\det(\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix}) = (a-\lambda)[(a-\lambda)^2 - b^2].$$

This leads to $\lambda_1 = a - b$, $\lambda_2 = a$, $\lambda_3 = a + b$.

2b: To find a diagonalizer, we need to compute eigenvectors. Note that the eigenvalues are distinct since $b \neq 0$.
For $\lambda_1 = a - b$, we need to solve

$$\begin{bmatrix} b & 0 & b \\ 0 & b & 0 \\ b & 0 & b \end{bmatrix} x_1 = 0.$$

This leads to

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = a$, we need to solve

$$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} x_2 = 0.$$

This leads to

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda_3 = a + b$, we need to solve

$$\begin{bmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{bmatrix} x_3 = 0.$$

This leads to

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To obtain an orthogonal diagonalizer, we only have to normalize the eigenvectors:

$$\begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a-b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a+b \end{bmatrix}.$$

A matrix M is

- *idempotent* if $M^2 = M$.
- *nilpotent* if $M^k = 0$ for some integer $k \geqslant 0$.
- *involutory* if $M^2 = I$.

(a) Show that every eigenvalue of an idempotent matrix is either 0 or 1.

(b) Show that every eigenvalue of a nilpotent matrix is 0.

(c) Show that every eigenvalue of an involutory matrix is either −1 or 1.

(d) Show that the only diagonalizable nilpotent matrix is the zero matrix.

(e) Show that every nonsingular idempotent matrix diagonalizes itself.

(f) Show that A is involutory if and only if $\frac{1}{2}(A + I)$ is idempotent.

(g) Show that every involutory matrix is diagonalizable.

Required Knowledge: **Eigenvalues, eigenvectors, nonsingularity, diagonalizability.**

Solution:

3a: Suppose that M is an idempotent matrix. Let (λ, x) be an eigenpair, that is $Mx = \lambda x$ and $x \neq 0$. Note that $M^2 x = \lambda^2 x$. Then, we have

$$0 = (M^2 - M)x = (\lambda^2 - \lambda)x.$$

This means that $\lambda^2 - \lambda = 0$ since $x \neq 0$. Therefore, either $\lambda = 0$ or $\lambda = 1$.

3b: Suppose that M is a nilpotent matrix. Let (λ, x) be an eigenpair, that is $Mx = \lambda x$ and $x \neq 0$. Note that $M^k x = \lambda^k x$ for any integer $k \geqslant 0$. Then, we have

$$0 = M^k x = \lambda^k x.$$

Consequently, $\lambda = 0$ since $x \neq 0$.

3c: Suppose that M is an involutory matrix. Let (λ, x) be an eigenpair, that is $Mx = \lambda x$ and $x \neq 0$. Note that $M^2 x = \lambda^2 x$. Then, we have

$$0 = (M^2 - I)x = (\lambda^2 - 1)x.$$

As such, we obtain that either $\lambda = -1$ or $\lambda = 1$.

3d: Suppose that $M^k = 0$ for some $k \geqslant 0$ and $M = TDT^{-1}$ where D is a diagonal matrix. Then, we have

$$0 = M^k = \underbrace{TDT^{-1}TDT^{-1} \cdots TDT^{-1}}_{k \text{ times}} = TD^kT^{-1}.$$

This means that $D^k = 0$ and hence $D = 0$. Therefore, $M = TDT^{-1} = 0$.

3e: Suppose that M is a nonsingular idempotent matrix. Observe that

$$M^{-1}MM = M^{-1}M^2 = M^{-1}M = I.$$

Therefore, we have $M = MIM^{-1}$. In other words, M diagonalizes itself.

3f: Note that

$$\begin{aligned}
\frac{1}{2}(A+I) \text{ is idempotent } &\iff \frac{1}{2}(A+I)\frac{1}{2}(A+I) = \frac{1}{2}(A+I) \\
&\iff \frac{1}{4}(A^2+2A+I) = \frac{1}{2}(A+I) \\
&\iff \frac{1}{4}(A^2-I) = 0 \\
&\iff A^2 = I \\
&\iff A \text{ is involutory.}
\end{aligned}$$

3g: Suppose that $M \in \mathbb{R}^{n\times n}$ is an involutory matrix. We know that M is diagonalizable if and only if it has n linearly independent eigenvectors. Suppose that M has at most $k < n$ linearly independent eigenvectors, say $u_1, u_2, \ldots, u_k$. Let $v \notin \text{span}(u_1, u_2, \ldots, u_k)$. We distinguish two cases:

- case 1: Mv is a multiple of v.

  In this case, v is an eigenvector of M. Since $v \notin \text{span}(u_1, u_2, \ldots, u_k)$, the set $\{u_1, u_2, \ldots, u_k, v\}$ is linearly independent. Hence, M has $k+1$ linearly independent eigenvectors. Contradiction!

- case 2: Mv is not a multiple of v.

  In this case, neither $v + Mv$ nor $v - Mv$ is zero. Note that

  $$\begin{aligned}
  M(v+Mv) &= Mv + M^2v = Mv + v = v + Mv \\
  M(v-Mv) &= Mv - M^2v = Mv - v = -(v - Mv).
  \end{aligned}$$

  This means that both $v+Mv$ and $v-Mv$ are eigenvectors of M. Since $v \notin \text{span}(u_1, u_2, \ldots, u_k)$, the vectors $v+Mv$ and $v-Mv$ cannot belong to $\text{span}(u_1, u_2, \ldots, u_k)$ at the same time. Hence, M has $k+1$ linearly independent eigenvectors. Contradiction!

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Find a singular value decomposition of A.

(b) Find the best rank 1 approximation for A.

REQUIRED KNOWLEDGE: **Singular value decomposition, lower rank approximation.**

SOLUTION:

4a: Note that

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the eigenvalues of $A^T A$ are given by

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 1$$

and hence the singular values of M are given by

$$\sigma_1 = \sqrt{2} \quad \text{and} \quad \sigma_2 = 1.$$

Note that

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are the normalised eigenvectors for λ_1 and λ_2, respectively. This results in

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$u_2 = \frac{1}{\sigma_2} M v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now, we need to find an orthonormal basis for $\mathcal{N}(A^T)$. Consider the linear system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} w = 0.$$

This leads to, for instance,

$$w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Then, we obtain

$$u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Thus, we have the SVD:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4b: The best rank 1 approximation is given by:

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
